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OPTIMAL DESIGN OF SYMMETRIC STRUCTURES AGAINST POSTBUCKLING COLLAPSE

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Abstract—Bars, trusses, beams, plates and other "strictly symmetric" structures have been shown previously to buckle under increasing loads, which often approach limiting values as buckling progresses. These collapse loads represent a more reasonable design criterion than the initial buckling loads since unavoidable imperfections often make the latter entirely meaningless, but do not affect extended postbuckling behavior.

The equations governing optimal design against collapse are developed and shown to be somewhat simpler than the equations governing optimum buckling strength. They are applied to the example of a fixed-fixed beam, which buckles laterally and torsionally and which collapses under a load which is about one third greater than its buckling load. This collapse load can in turn be increased substantially if optimal rather than prismatic design is selected.

1. INTRODUCTION

Optimal design of columns against buckling was first treated in detail by Keller[1] and by Tadjbakhsh and Keller[2]. Different aspects of the problem have since received the attention of many investigators (e.g. [3, 4]), while the question of absolute vs relative optimality has very recently been reexamined and corrected by Olhoff and Rasmussen[5]. Popelar[6, 7], in studying optimal design of beams against lateral buckling, has arrived at the unexpected result that for constant bending moment the prismatic beam is also best. The literature abounds with other examples.

In general these examples have the common feature that the "primary stresses," which prevail in the unbuckled state and which "cause" the instability, are statically determinate and are therefore known *a priori* independently of the design. For such structures the best design can be obtained from basic energy principles[3-7], and it has been shown[8] that the best design is in general the one which exhibits constant average strain energy in the "design fibers," that is, the critical fibers which are affected by a change in the design.

No such principles appear to exist if the structure is statically indeterminate and if, as a result, the primary stresses themselves are affected by a change in the design. In the general case this condition poses substantial computational difficulties, especially if the structure is imperfection sensitive and exhibits unstable points of bifurcation.

Certain types of structures may be called "strictly symmetric" or "completely symmetric"[9], and these structures exhibit a different type of postbuckling behavior. They are characterized by the fact that in the potential energy expansion in the neighborhood of the critical bifurcation point the cubic term vanishes identically for all assumed buckling modes. Such structures have been shown[9] to have stable points of bifurcation, to buckle under increasing loads, and, in certain cases, to approach limiting collapse configurations associated with finite loading parameters. Examples of this type of structure are trusses which are statically indeterminate with respect to their axial forces[10], or statically indeterminate beams buckling laterally[11]. The theory on which the predicted collapse loads are based is essentially an "intermediate" theory (similar to the Karman theory of plates). Recently published large deflection theories[12, 13] have confirmed the predictions of the intermediate theory through most of the range of deflection amplitudes. Experimental confirmation has been supplied in [14].

The significance of the collapse load as a reasonable design criterion, in contrast to that of the initial buckling load, is enhanced by the elimination of initial imperfections as a factor in determining either the magnitude of the collapse load or the nature of the collapse mode. This is, of course, not surprising. What may be surprising, however, is the fact that the collapse conditions may also be unaffected by certain boundary conditions relating to the primary stresses. For example, the lateral buckling load of a beam which is elastically constrained in its major plane of stiffness at the supports is strongly influenced by the stiffness of those constraints; nevertheless, its collapse value is independent of the (nonvanishing) constraints, and the design of a beam against collapse may therefore proceed before the stiffness of the constraints has been established.

In what follows we present a brief derivation of the general postbuckling and collapse conditions of strictly symmetric structures, and we establish the conditions which must be satisfied if a structure of given volume is to exhibit the largest possible collapse load. We then apply the general formulation to the case of a beam which is fixed (or at least, as pointed out before, elastically constrained) in its major plane of stiffness at the supports and which collapses by a combination of lateral buckling and rotation. For the case of a centrally-loaded prismatic beam, the value of the load increases more than thirty percent between initial buckling and collapse. It is shown that optimal design adds a significant increase to this collapse value.

2. POSTBUCKLING AND COLLAPSE

Inherent in the concept of a strictly symmetric structure is the idea that the displacements can be split into two categories. The first, designated by u, represents the primary displacements, associated with the primary strains ϵ and stresses σ, which already occur in the unbuckled configuration. To this we add the buckling displacements v, with associated strains κ and stresses m. For example, in the case of classical plate theory the first category covers in-plane displacements, strains and forces, the second the lateral displacement, curvatures and bending moments. The kinematic relations are given by

$$\epsilon = L_1(u) + 1/2L_2(v) \tag{1}$$

$$\kappa = k(v), \tag{2}$$

in which L_1 and k are linear in their arguments and L_2 is quadratic. The elastic constitutive relations are

$$\epsilon = C\sigma \qquad (C = C^T) \tag{3}$$

$$m = K\kappa \qquad (K = K^T) \tag{4}$$

with C representing the primary compliance and K the buckling stiffness. Both matrices C and K are functions of the design parameter h, and both are symmetric.

The primary σ can, in general, be expressed by

$$\sigma = \lambda\sigma_o + \sigma', \tag{5}$$

in which λ is a load parameter and σ' represents a stress field satisfying homogeneous equations of equilibrium and boundary conditions. If the class of all self-equilibrated stress fields is designated by $[\sigma_e]$, then it follows from the principle of virtual work that

$$[\sigma_e{}^T]L_1(u) = 0, \tag{6}$$

in which the notation implies summation or integration over the entire structure. With the substitution of eqns (1) and (3) this equation becomes

$$[\sigma_e{}^T][C\sigma - 1/2L_2(v)] = 0 \tag{7}$$

representing the condition of primary compatibility. Let $\lambda\sigma_o$ be the primary stress in the unbuckled structure, that is, $\sigma' = 0$ when $v = 0$. Substitution in eqn (7) then leads to the familiar

$$[\sigma_e{}^T]C\sigma_o = 0, \tag{8}$$

and this, in turn, converts eqn (7) into the alternate compatibility relationship

$$[\sigma_e{}^T][C\sigma' - 1/2L_2(v)] = 0. \tag{9}$$

Equilibrium with respect to the buckling displacements, in variational form, is given by

$$m^T\delta\kappa + \sigma^T L_{11}(v, \delta v) = 0. \tag{10}$$

With eqns (2) and (4) this takes the form of

$$k^T(v)Kk(\delta v) + \sigma^T L_{11}(v, \delta v) = 0. \tag{11}$$

The postbuckling state is now determined by eqns (9) and (11). We note that the former contains the design variable h through C, and this is the reason why a change in the design also affects the primary stress system σ_0. However, if we assume, in the limit,

$$\begin{aligned} &v = \omega v_c \qquad (\omega \to \infty) \\ &\lambda \to \lambda_c < \infty \qquad \sigma' \to \sigma_c \quad (<\infty), \end{aligned} \tag{12}$$

that is, if a finite primary stress and load parameter is reached for increasing buckling amplitude, then eqn (9) and (11) become, respectively,

$$[\sigma_e{}^T]L_2(v_c) = 0 \tag{13}$$

$$k^T(v_c)Kk(\delta v) + (\lambda_c\sigma_o + \sigma_c)^T L_{11}(v_c, \delta v) = 0. \tag{14}$$

These equations no longer depend on the compliance C, and the optimization process becomes correspondingly simpler. It is interesting to observe that this simplification is analogous to the theory of plastic optimality, which is also simpler than elastic optimality because of the removal of the compatibility restriction[15].

3. OPTIMALITY

The governing collapse eqns (13) and (14) are identically satisfied for all designs $h(x)$. A variation of the design (identified by a superimposed dot) therefore leads to the system of equations

$$[\sigma_e{}^T]L_{11}(v_c, \dot{v}_c) = 0 \tag{15}$$

$$k^T(\dot{v}_c)Kk(\delta v) + (\lambda_c\sigma_o + \sigma_c)^T L_{11}(\dot{v}_c, \delta v) = -k^T(v_c)\frac{\mathrm{d}K}{\mathrm{d}h}k(\delta v)\dot{h} - (\dot{\lambda}_c\sigma_o + \dot{\sigma}_c)^T L_{11}(v_c, \delta v). \tag{16}$$

If only designs of the same volume V are considered, then $h(x)$ is restricted by

$$\dot{V} = \int \frac{\mathrm{d}A}{\mathrm{d}h}\dot{h}\,\mathrm{d}x = 0. \tag{17}$$

Also for optimality,

$$\dot{\lambda}_c = 0. \tag{18}$$

The left side of eqn (16) contains a linear homogeneous operator in $\dot{v}_c$. The same operator is shown in eqn (14) to be singular, and eqn (16) therefore has no solution unless the secular term on the right side is removed. This is achieved by subtracting eqn (14) (with $\delta v = \dot{v}_c$) from eqn (16) (with $\delta v = v_c$). The restriction of eqn (17) is removed through the introduction of the Lagrangian multiplier μ^2. In view of eqns (18) and (13) (since $\dot{\sigma}_c \epsilon [\sigma_e]$), this leads to the optimality condition

$$\int \left[k^T(v_c)\frac{\mathrm{d}K}{\mathrm{d}h}k(v_c) - \mu^2\frac{\mathrm{d}A}{\mathrm{d}h}\right]\dot{h}\,\mathrm{d}x = 0. \tag{19}$$

Equation (19) is in global form and may have to be used if h is subject to constraints, such as are imposed in the example introduced in the next section. For arbitrary variation $\dot{h}$ we obtain

$$k^T(v_c)\frac{\mathrm{d}K}{\mathrm{d}h}k(v_c)-\mu^2\frac{\mathrm{d}A}{\mathrm{d}h}\equiv 0, \tag{20}$$

which shows once again that optimality corresponds to a prescribed distribution of the energy in the design fibers. In the presence of design constraints of the type $h(x)\geqq h_{\min}$, eqn (20) covers that portion of the structure in which the constraints are not active.

Equation (20) is a necessary condition for the collapse load to be stationary. Its sufficiency is demonstrated by considering

$$\frac{\dot{\lambda}_c}{\lambda_c}=\frac{\int k^T(v_c)\frac{\mathrm{d}K}{\mathrm{d}h}k(v_c)\dot{h}\,\mathrm{d}x}{\int k^T(v_c)Kk(v_c)\,\mathrm{d}x}, \tag{21}$$

which, with the substitution of eqns (20) and (17), leads to eqn (18).

We note that the governing eqns (13) (collapse), (14) (equilibrium) and (20) (optimality) are derivable by introducing a single functional Ω such that

$$\delta\left(\Omega-\mu^2\int A(h)\,\mathrm{d}x\right)=0$$

where

$$\Omega(\lambda_c;\sigma_c,v_c,h)\equiv\frac{1}{2}\int[k^T(v_c)K(h)k(v_c)+(\lambda_c\sigma_o+\sigma_c)^TL_2(v_c)]\,\mathrm{d}x \tag{22}$$

and where the variations are taken independently with respect to σ_c, v_c and h. The solution of these equations, however, is not unique since they apply to any of the buckling branches of the structure. Stability as well as uniqueness require that the solution lie on the lowest branch, that is, that

$$\Omega(\lambda_c;\sigma_c,v,h)\geqq 0 \tag{23}$$

for all v, with the equality applying to the actual collapse mode v_c.

Under certain circumstances eqn (20) is sufficient for global (rather than relative) optimality. In fact, consider two designs $h_1(x)$ and $h_2(x)$ satisfying eqns (13) and (14) for $v_c=v_1$ and $v_c=v_2$, $\lambda_c=\lambda_1$ and $\lambda_c=\lambda_2$ and $\sigma_c=\sigma_1$ and $\sigma_c=\sigma_2$, respectively. Also, let h_1 be optimal in the sense of eqn (20). Then (again deleting integral signs), by eqn (23),

$$k^T(v_1)K(h_1)k(v_1)+\lambda_1\sigma_o{}^TL_2(v_1)=0 \tag{24}$$

$$k^T(v_2)K(h_2)k(v_2)+\lambda_2\sigma_o{}^TL_2(v_2)=0 \tag{25}$$

and, again by eqn (23),

$$k^T(v_1)K(h_2)k(v_1)+\lambda_2\sigma_o{}^TL_2(v_1)\geqq 0. \tag{26}$$

Comparison of eqns (24) and (26) yields

$$k^T(v_1)[K(h_2)-K(h_1)]k(v_1)+(\lambda_2-\lambda_1)\sigma_o{}^TL_2(v_1)\geqq 0. \tag{27}$$

Let us now assume that $K(h)$ is concave in the sense that

$$\left[k^T\frac{\mathrm{d}K(h_1)}{\mathrm{d}h}k\right](h_2-h_1)\geqq k^T[K(h_2)-K(h_1)]k \tag{28}$$

for all v, and that $A(h)$ satisfies the convexity condition

$$A(h_2) - A(h_1) \geqq \frac{dA(h_1)}{dh}(h_2 - h_1). \tag{29}$$

Then the inequality (27), in view of eqns (28) and (29) and by considering the optimality condition eqn (20), becomes

$$\mu^2 \int [A(h_2) - A(h_1)]\, dx + (\lambda_2 - \lambda_1) \int \sigma_o{}^T L_2(v_1)\, dx \geqq 0. \tag{30}$$

The second integral is negative in view of eqn (24), of the positive definiteness of K, and by postulating, without loss of generality, that $\lambda_1 > 0$. It then follows from eqn (30) either that for given load ($\lambda_2 = \lambda_1$) the design h_1 corresponds to minimum volume, or, equivalently, that for given volume the (positive) load is a maximum.

It is important to emphasize once again that the range of applicability of eqn (20) as a sufficient condition for global optimality is limited. It covers, above all, the important case in which K and A are linear functions of h. On the other hand, it does not cover the equally important case of $K = K_0 h^n$ for $n > 1$; this shortcoming may become critical in statically indeterminate cases, as has recently been demonstrated by Olhoff and Rasmussen[5], who have shown the solution found in [2] to represent a local rather than a global optimum.

In analogy to the theory of perfect plasticity (and in fact borrowing its terminology), the actual collapse load λ_c may be bracketed between a class of "statically admissible" load parameters $[\lambda_s]$ and "kinematically admissible" load parameters $[\lambda_k]$. The definition of the former requires that there exist a stress field $\sigma_s \epsilon [\sigma_e]$ such that

$$\Omega(\lambda_s; \sigma_s, v, h) \geqq 0 \quad \text{for all } v. \tag{31}$$

The latter is based on any collapse mechanism v_k satisfying eqn (13), provided that there exists a stress field $\sigma_k \epsilon [\sigma_e]$ such that

$$\Omega(\lambda_k; \sigma_k, v_k, h) \leqq 0. \tag{32}^\dagger$$

If eqn (32) is subtracted from eqn (31) (with $v = v_k$), then, in view of eqn (13),

$$(\lambda_s - \lambda_k)\sigma_o{}^T L_2(v_k) \geqq 0. \tag{33}$$

For $\lambda_k > 0$ the integral expression in eqn (33) is again negative. Moreover, since the actual collapse load parameter λ_c is both statically and kinematically admissible, it follows that

$$\lambda_s \leqq \lambda_c \leqq \lambda_k. \tag{34}$$

Again in analogy with perfect plasticity the inequalities (34) imply that if, for given volume, the optimization process is based entirely on the static (kinematic) method, then the result represents a lower (upper) bound to the optimal collapse load. Conversely, for given collapse load, the two methods lead to upper (lower) bounds to the optimum volume. Practically speaking, only the static method appears to be computationally advantageous.

4. EXAMPLE

As an example to demonstrate the application of the general equations of the previous section, we analyze the lateral buckling of a beam, of length $2l$, which is subjected to a concentrated load P at the center and which is fixed (or, as outlined in the Introduction, partially restrained) in its major plane at the two supports. The beam is also assumed to be simply supported relative to its lateral deflection.

†The satisfaction of eqns (13) and (32) is sufficient to assure the existence of a collapse parameter λ_c.

The beam is of rectangular cross section of constant width b, the design variable being the depth $h(x)$. Because of symmetry only the left half of the beam $(0 \leqq x \leqq l)$ nned be considered. With

$$\left.\begin{aligned} &A = bh \\ &K = K_o h \qquad I = I_0 h \\ &V = \int_o^l bh\,\mathrm{d}x \equiv blh_o \\ &P = 2\lambda \frac{h_o}{l^2} \surd(EI_o GK_o) \end{aligned}\right\} \tag{35}$$

in which K and I represent, respectively, the torsional and weak bending stiffnesses (K_o and I_o being constants), we note that eqns (28) and (29) and hence the conditions for global optimality are satisfied. The place of the primary stress σ is taken by the primary bending moment $m(x)$, which is given by

$$m = \lambda \frac{h_o}{l} \surd(EI_o GK_o)\left(\frac{x}{l} - \alpha\right), \quad (0 \leqq x \leqq l) \tag{36}$$

in which α accounts for the effect of the redundant support moment.

Since warping may be neglected the problem is governed by the functional

$$\Omega' = \frac{1}{2}\int_o^l (EIu''^2 + GK\beta'^2 - 2mu''\beta - 2\mu^2 GK_o h)\,\mathrm{d}x \tag{37}$$

subject to the boundary conditions

$$\beta(0) = \beta'(l) = EIu''(0) = (EIu'' - m\beta')'_{(l)} = 0, \tag{38}$$

in which $u(x)$ and $\beta(x)$ represent the lateral displacement and rotation, respectively, and a prime designates differentiation. Variation with respect to u, two integrations, and consideration of the boundary conditions lead to

$$EIu'' - m\beta = 0 \qquad (0 \leqq x \leqq l), \tag{39}$$

which, after insertion into eqn (37), together with eqn (36), leads to

$$\Omega = \frac{1}{2}\int_o^l \left[h\beta'^2 - \lambda^2 \frac{h_o^2}{hl^2}\left(\frac{x}{l} - \alpha\right)^2 \beta^2 - 2\mu^2 h\right]\mathrm{d}x \tag{40}$$

together with the first two of the boundary conditions in eqn (38). We note that Ω is now a functional of $\beta(x)$, $h(x)$, λ and α.

The equilibrium equation follows from variation of Ω with respect to β (with the subscript c deleted):

$$(h\beta')' + \lambda^2 \frac{h_0^2}{l^2}\left(\frac{x}{l} - \alpha\right)^2 \frac{\beta}{h} = 0 \qquad (0 \leqq x \leqq l)$$

$$\beta(0) = \beta'(l) = 0. \tag{41}$$

The collapse condition, that is, $\delta_\alpha \Omega = 0$, is given by

$$\int_o^l \left(\frac{x}{l} - \alpha\right)\frac{\beta^2}{h}\,\mathrm{d}x = 0, \tag{42}$$

while the optimality condition $\delta_h \Omega = 0$ results in

$$(\beta')^2 - \lambda^2 \frac{h_o^2}{l^2}\left(\frac{x}{l} - \alpha\right)^2 \frac{\beta^2}{h^2} = \mu^2 \qquad (0 \leqq x \leqq l). \tag{43}$$

Alternatively, substitution of eqn (41) in eqn (43) leads to the somewhat simplified form of the optimality equation

$$(\beta')^2 - \frac{\beta}{h}(h\beta')' \equiv -\frac{\beta^2}{h}\left(\frac{h\beta'}{\beta}\right)' = \mu^2 \qquad (0 \leqq x \leqq l) \tag{43'}$$

which, when multiplied by h and integrated over the half length of the beam, identifies the Lagrangian multiplier μ^2 by

$$\mu^2 = \frac{2}{h_o l}\int_o^l h(\beta')^2\,\mathrm{d}x \tag{44}$$

after integration by parts and in view of eqn (35) and the boundary conditions.

For the problem under consideration the equivalent of eqn (21) is

$$\frac{\dot{\lambda}_c}{\lambda_c} = -\frac{\int_o^l \frac{\beta^2}{h}\left(\frac{h\beta'}{\beta}\right)'\dot{h}\,\mathrm{d}x}{2\int_o^l h(\beta')^2\,\mathrm{d}x}. \tag{45}$$

This suggests an iterative solution procedure. Assume a design $h = h_1(x)$ associated with $\beta = \beta_1(x)$ and μ_1, and let a new design be given by

$$h_2 = h_1 + \dot{h}$$

where

$$\dot{h}(x) = -\epsilon\left[h_1 + \frac{\beta_1^2}{\mu_1^2}\left(\frac{h_1\beta_1'}{\beta_1}\right)'\right], \qquad (0 < \epsilon \leqq 1) \tag{46}$$

which by eqn (44) can readily be shown to satisfy the condition of equal volume. If $\dot{h}$ as given in eqn (46) is inserted into eqn (45), then, after some algebra,

$$\frac{1}{\epsilon}\mu_1^4\frac{\dot{\lambda}_c}{\lambda_c} = \int_o^l \frac{1}{h_1}\left[\beta_1^2\left(\frac{h_1\beta_1'}{\beta_1}\right)'\right]^2\mathrm{d}x - \frac{\left[\int_o^l \beta_1^2\left(\frac{h_1\beta_1'}{\beta_1}\right)'\mathrm{d}x\right]^2}{\int_o^l h_1\,\mathrm{d}x} \geqq 0 \tag{47}$$

in which the inequality is that of Schwarz. In other words, repetitive design, separated by repetitive analysis, leads to a monotonically increasing sequence of load values. Since the ultimate load parameter is bounded, and if ϵ is small enough, the sequence therefore converges.

Numerical analysis

The governing eqns (41)–(43) or (43′) have been solved numerically by introducing the finite element method. Accordingly it is assumed that the depth h over each element is constant; actually, this is a more realistic approach from the point of view of practical design procedure than is the assumption of continuously varying depth. With the customary approximate assumption of a linear shape function for $\beta(x)$ and constant bending moment $m(x)$ over the element, that is, with

$$\left.\begin{aligned} h(x) &= h_o\eta_i \\ \beta(x) &= \frac{x_i - x}{a_i}\beta_{i-1} + \frac{x - x_{i-1}}{a_i}\beta_{i-1} \qquad (x_{i-1} \leqq x \leqq x_i) \\ m(x) &= 1/2[m(x_{i-1}) + m(x_i)] \qquad (i = 1, 2, \ldots, n). \\ a_i &= x_i - x_{i-1} \end{aligned}\right\} \tag{48}$$

Equation (40) now becomes

$$\Omega = \frac{h_o}{2}\sum_{k=1}^{n}\frac{1}{a_k}\left\{\eta_k(\beta_k-\beta_{k-1})^2-\frac{\lambda^2 a_k^2}{3\eta_k l^2}\gamma_k^2(\beta_{k-1}^2+\beta_{k-1}\beta_k+\beta_k^2)-2\mu^2 a_k^2\eta_k\right\} \tag{49}$$

where

$$\gamma_k \equiv \frac{1}{2l}(x_{k-1}+x_k)-\alpha.$$

Equation (49) is the basis for the following system of equations:

(1) $\partial\Omega/\partial\beta_i = 0$ ($i = 1, 2, \ldots, n-1$) leads to ($n-1$) equations of equilibrium which are linear homogeneous in β_i and contain the eigenvalue λ^2.

(2) $\partial\Omega/\partial\alpha = 0$ corresponds to one equation of collapse.

(3) $\partial\Omega/\partial\eta_i = 0$ ($i = 1, 2, \ldots, n$) constitutes n equations of optimality.

(4) $\sum_{k=1}^{n}\eta_k$ = constant takes care of the volume constraint.

These $2n+1$ equations contain the $2n+2$ unknowns β_i, η_i, α, λ, μ; however, because of the homogeneity of the equations in β_i the amplitude of the latter is undetermined, and the problem is well posed, though nonlinear.

For the solution of the problem the beam was divided into six elements of equal length. For the prismatic case ($\eta_i = 1$) initial buckling occurs when $\alpha = 0.5000$, which is associated with a load parameter $\lambda = 5.788$. At collapse $\alpha = 0.6563$ and $\lambda = 7.612$ representing an increase of 31.5% over the initial buckling value. The optimum design is reached when $\eta_i = 1.7859, 0.9918, 0.2223$ (and symmetric values for the right half of the beam). This corresponds to $\alpha = 0.7801$ (representing a further shift from the positive center moment to the negative support moment) and a collapse value of $\lambda = 8.907$. In other words, optimal design improves the collapse strength by 17% over prismatic design.

It may be noted that the buckling mode β_i itself is affected very little by either proceeding from initial buckling to collapse for the prismatic beam, or by redesigning the beam for optimal collapse strength. This suggests that estimates may be obtained with relatively little labor by means of the traditional numerical techniques of the Ritz–Galerkin type.

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